①

The Eisenstein ideal of weight k and ranks of Hecke algebras

Let $p>3$ be a prime, ℓ be a prime s.t. $p \mid \ell-1$ and $k \geq 2$ be an even integer

$M_k(\ell, \mathbb{Z}_p)$ = Space of modular forms of wt k, level $\Gamma_0(\ell)$ with Fourier coeff. in $\mathbb{Z}_p$.

$$\underset{q\text{ exp. principle}}{\subseteq} \mathbb{Z}_p[\![q]\!]$$

$S_k(\ell, \mathbb{Z}_p)$ = Subspace of $M_k(\ell, \mathbb{Z}_p)$ consisting of cusp forms $\underset{q\text{ exp.}}{\subseteq} \mathbb{Z}_p[\![q]\!]$

<u>E.g.</u> → 1) Let $k>2$ even and assume $(p-1) \nmid k$

Then

$$E_k(q) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n$$

$$\in M_k(\ell, \mathbb{Z}_p)$$

$$\sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1}$$

$$E_k(q^\ell) \in M_k(\ell, \mathbb{Z}_p)$$

②

2) For $k=2$

$$E_2(q) = \frac{-1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

Then $E_2(q) - l E_2(q^l) \in M_2(l, \mathbb{Z}_p)$

3) $\Delta = \sum_{n=1}^{\infty} \tau(n) q^n \in S_{12}(l, \mathbb{Z}_p)$

$\uparrow$ Ramanujan's tau fn.

For every prime $l' \neq l$, we have

a Hecke operator $T_{l'} \curvearrowright M_k(l, \mathbb{Z}_p)$

and the action also stabilizes $S_k(l, \mathbb{Z}_p)$

$\Rightarrow$ $T_{l'} \in S_k(l, \mathbb{Z}_p)$

There is also Atkin-Lehner involution

$w_l \curvearrowright M_k(l, \mathbb{Z}_p)$, $S_k(l, \mathbb{Z}_p)$

If $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(l, \mathbb{Z}_p)$

$$T_{l'} f = \sum_{n=0}^{\infty} a_{nl'} q^n + l'^{k-1} \sum_{n=0}^{\infty} a_n q^{nl'}$$

③

For all prime $\ell' \neq \ell$, $W_\ell T_{\ell'} = T_{\ell'} W_\ell$

and for all primes ℓ', ℓ'', $T_{\ell'} T_{\ell''} = T_{\ell''} T_{\ell'}$

Let $\mathbb{T} = \mathbb{Z}_p$-subalg. of $\mathrm{End}_{\mathbb{Z}_p}(M_k(\ell, \mathbb{Z}_p))$
gen. by $T_{\ell'}$ for all primes $\ell' \neq \ell$
and W_ℓ.

$\mathbb{T}^0 =$ Quotient of $\mathbb{T}$ which acts faithfully on $S_k(\ell, \mathbb{Z}_p)$

$f \in M_k(\ell, \mathbb{Z}_p)$ then f is called an eigenform if it is an eigenvector for all $T_{\ell'}$'s and W_ℓ

E.g. → If $k \geq 2$ even, and $(p-1) \nmid k$

Then $E_k(q) - \ell^{k/2} E_k(q^\ell) \in M_k(\ell, \mathbb{Z}_p)$

is an eigenform with
$T_{\ell'}$ - eigenvalue $= 1 + (\ell')^{k-1}$
W_ℓ - eigenvalue $= -1$

I_{eis} = Ideal of $\mathbb{T}$ gen. by the set

$\{T_{\ell'} - (1 + (\ell')^{k-1}) \mid \ell' \neq \ell \text{ prime}\}$

$\cup \{W_\ell + 1\}$ = Eisenstein ideal of wt k

④

i.e. $\phi : \mathbb{T} \to \mathbb{Z}_p$

$T_{\ell'} \mapsto T_{\ell'}$-Eigenvalue of $E_k - \ell^{k/2} E_k(q^\ell)$

$w_\ell \mapsto -1$

then ϕ is a ring hom and $\ker(\phi) = I^{eis}$

$\mathbb{T} / I^{eis} = \mathbb{Z}_p$

Let m = max^{l} ideal of $\mathbb{T}$ gen. by p and I^{eis}

Congruences of eigenforms →

Suppose $\mathcal{O}$ is a ring of int. of a finite ext^n of $\mathbb{Q}_p$ and let π be the unif. of $\mathcal{O}$

Let $f \in M_k(\ell, \mathcal{O})$ be an eigenform

Then we say that $f \equiv E_k \mod (p)$

if $T_{\ell'}$ eigenvalue of $f \equiv T_{\ell'}$ eigenvalue of E_k $(\text{mod } \pi)$

$\forall$ primes $\ell' \neq \ell$

If f is normalized i.e. $f(q) = q + \sum_{n=2}^{\infty} a_n q^n$

then $f \equiv E_k \mod (p)$

$\iff$ $a_n(f) \equiv a_n(E_k) \mod p$ for all $\ell \nmid n$

⑤

Reformulation in terms of Galois repⁿ

Deligne, Eichler-Shimura →

$\exists$ a continous, odd, irred. repⁿ

$$\rho_f : G_{\mathbb{Q}, p\ell} \rightarrow GL_2(\mathcal{O}) \quad \text{for, primes } \ell' \neq \ell, p$$

s.t. $tr(\rho_f(Frob_{\ell'})) = T_{\ell'}$ - eigenvalue of f

$\det(\rho_f) = \chi_p^{k-1}$ ← p-adic cycl. char.

Let $\bar{\rho}_f : G_{\mathbb{Q}, p\ell} \rightarrow GL_2(\mathcal{O}) \xrightarrow{\bmod \pi} GL_2(\mathbb{F})$ ← fin extⁿ of $\mathbb{F}_p$

Then $f \equiv E_k \pmod{p}$

$\Leftrightarrow \bar{\rho}_f^{ss} = 1 \oplus \omega_p^{k-1}$

ω_p mod p cyclo. char.

Fact → $\exists$ a p-ordinary eigenform $f \in S_k(\ell, \mathcal{O})$ (for some $\mathcal{O}$), with W_ℓ eigenvalue -1

s.t. $f \equiv E_k \pmod{p}$

(p-ordinary means T_p-eigenvalue of f is a p-adic unit)

- $k = 2$ Due to Mazur
- $k > 2$ Billerey-Menares, Wake

⑥

$\Rightarrow$ The image of $\underset{\substack{\| \\ (p, I^{eis})}}{m}$ in $\mathbb{T}^0$ is a maximal ideal of $\mathbb{T}^0$ which we denote by m^0.

Let $\mathbb{T}_m$ = Completion of $\mathbb{T}$ at m
$\mathbb{T}^0_m$ = Completion of $\mathbb{T}^0$ at m^0

Fact $\Rightarrow$ $\mathrm{rank}_{\mathbb{Z}_p}(\mathbb{T}^0_m) \geq 1$

Qn $\rightarrow$ What can be said about $\mathrm{rank}_{\mathbb{Z}_p}(\mathbb{T}^0_m)$?.

(i.e. how many cusp forms of wt k and level $\Gamma_0(\ell)$ are congruent to E_k mod $p\ell$.)

For $k=2$ This was asked by Mazur

History $\rightarrow$ $\boxed{k=2}$

"Analytic Side"

Thm (Merel) $\rightarrow$ $\mathrm{rank}_{\mathbb{Z}_p}(\mathbb{T}_m^0) > 1$

$\Leftrightarrow$ The image of $\prod_{i=1}^{\frac{\ell-1}{2}} i^i$ in $(\mathbb{Z}/\ell\mathbb{Z})^\times$ is a p-th power. (Some results for $p=2,3$)

Lecouturier has generalized this to give similar equiv. for rank >2

7

"Algebraic Side"

<u>Thm ((Calegari - Emerton) :</u>

If p-part of class group of $\mathbb{Q}(\ell^{1/p})$ is cyclic then $\mathbb{T}_m^0 \simeq \mathbb{Z}_p$

(+ some results $p = 2, 3$)

Let a_0 be a gen of

Inertia at p

$$\ker(H^1(G_{\mathbb{Q},p\ell}, 1) \to H^1(I_p, 1))$$

c_0 be a gen of

$$\ker(H^1(G_{\mathbb{Q},p\ell}, \omega_p^{-1}) \to H^1(I_p, \omega_p^{-1}))$$

$$\Rightarrow c_0 \cup a_0 \in H^2(G_{\mathbb{Q},p\ell}, \omega_p^{-1})$$

cup prod.

K_0 = Compositum of $\mathbb{Q}(S_p)$ and unique deg. p extn of $\mathbb{Q}$ contained in $\mathbb{Q}(S_\ell)$

<u>Thm (Wake - Wang - Erickson) :</u> TFAE

1) $\mathbb{T}_m^0 \simeq \mathbb{Z}_p$
2) $c_0 \cup a_0 \neq 0$
3) $\dfrac{Cl(K_0)}{Cl(K_0)^p}[\omega_p^{-1}]$ is cyclic.

8

<u>Aim</u> → Can we get similar criteria for $k>2$.

<u>Assume</u> → ① $(p-1) \nmid k$

② $\frac{Cl(\mathbb{Q}(\zeta_p))}{Cl(\mathbb{Q}(\zeta_p))^p}[\omega_p^{2-k}] = 0$

③ $\dim_{\mathbb{F}_p}(H^1(G_{\mathbb{Q},p}, \omega_p^{k-1})) = 1$

If p is regular prime ⇒ ② and ③ are satisfied.

Choose gen.

$c_0 \in \ker(H^1(G_{\mathbb{Q},p\ell}, \omega_p^{1-k}) \to H^1(I_p, \omega_p^{1-k}))$

$a_0 \in \ker(H^1(G_{\mathbb{Q},p\ell}, 1) \to H^1(I_p, 1))$

$b_0 \in H^1(G_{\mathbb{Q},p}, \omega_p^{k-1})$

$c_0 \cup b_0 \in H^2(G_{\mathbb{Q},p\ell}, 1)$, $c_0 \cup a_0 \in H^2(G_{\mathbb{Q},p\ell}, \omega_p^{1-k})$

<u>Thm (D)</u> → If $k>2$, TFAE

① $\mathbb{T}_m^0 \simeq \mathbb{Z}_p$

② $c_0 \cup a_0 \neq 0$, $c_0 \cup b_0 \neq 0$

9

(3) $\dfrac{Cl(K_0)}{Cl(K_0)^p}[\omega_p^{1-k}]$ is cyclic

and $\mathrm{Res}: H^1(G_{\mathbb{Q},p}, \omega_p^{k-1}) \to H^1(G_{\mathbb{Q}_\ell}, \omega_p^{k-1})$

is non-zero

(4) $\prod_{i=1}^{p-1} (1-\zeta_p^i)^{i^{k-2}} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ is not a p-th power

and $\prod_{i=1}^{\ell-1} i^{\left(\sum_{j=1}^{i-1} j^{k-1}\right)} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ is not a p-th power.

"Proof"

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) "Standard"

STP: (1) $\Leftrightarrow$ (2)

Gluing all Galois rep^{n}s attached to cuspforms which are congruent to E_k mod p

and get $\rho : G_{\mathbb{Q},p\ell} \to GL_2(\mathbb{T}_m^0)$

s.t. (1) $\det(\rho) = \chi_p^{k-1}$, $\mathrm{Tr}(\mathrm{Frob}_{\ell'}) = T_{\ell'}$ $\forall \ell' \neq \ell, p$

(2) $\mathrm{tr}(\rho(I_\ell)) = 2$

I_ℓ: inertia at ℓ

10

③ $\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$ where η_2 is unram

(ordinary at p)

④ $\bar{\rho} = \rho \pmod{m^\circ} = \begin{pmatrix} 1 & * \\ & \omega_p^{k-1} \end{pmatrix}$ $*$ corresponds to c_0

$\Rightarrow \exists$ a surj map $\phi : R_{\bar{\rho}} \to \mathbb{T}^\circ_m$

the ring which parametrizes all lifts of $\bar{\rho}$ satisfying the 4 properties above.

Suppose $c_0 \cup b_0 \neq 0 \Rightarrow I^{eis}$ is principal

and $\dim(\tan(R_{\bar{\rho}}/(p))) \leq 1$

$\Rightarrow R_{\bar{\rho}} \simeq \dfrac{\mathbb{Z}_p[\![x]\!]}{I}$

$\dim(\tan(R_{\bar{\rho}}/(p))) = 1 \Leftrightarrow \exists$ an ordinary lift

$\tilde{\rho} : G_{\mathbb{Q},pN} \to GL_2(\mathbb{F}[\varepsilon]/\varepsilon^2)$ of $\bar{\rho}$

s.t. $\tilde{\rho} = \begin{pmatrix} \tilde{\chi}_1 & * \\ & \tilde{\chi}_2 \end{pmatrix}$ with $\tilde{\chi}_1 \neq 1$, $\tilde{\chi}_2 \neq \omega_p^{k-1}$

11

$\Leftrightarrow\ c_0 \cup a_0 = 0$

$$\phi : R_{\bar\rho} \twoheadrightarrow \mathbb{T}_m^0$$

$R_{\bar\rho} = \dfrac{\mathbb{Z}_p[\![X]\!]}{I}$

You can choose X s.t. $\phi(X) =$ image of I^{eis} in $\mathbb{T}_m^0$ $:= I_0^{eis}$

Wake → $|\mathbb{T}_m^0 / I_0^{eis}| = p^{v + v_p(k)}$ $\quad v = v_p(\ell - 1)$

On the other hand, $|R_{\bar\rho}/(X)| \leq p^{v + v_p(k)}$

So $\mathbb{T}_m^0 \simeq \mathbb{Z}_p \Leftrightarrow \ker(\phi) = (X - p^{v + v_p(k)})$

$$\Leftrightarrow \mathbb{T}_m^0 \simeq \mathbb{Z}_p \simeq R_{\bar\rho}$$

$\Rightarrow \mathbb{T}_m^0 \simeq \mathbb{Z}_p \Leftrightarrow c_0 \cup a_0 \neq 0.$